

A two-mass expanding exact space-time solution

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Abstract

In order to understand how locally static configurations around gravitationally bound bodies can be embedded in an expanding universe, we investigate the solutions of general relativity describing a space-time whose spatial sections have the topology of a 3-sphere with two identical masses at the poles. We show that Israel junction conditions imply that two spherically symmetric static regions around the masses cannot be glued together. If one is interested in an exterior solution, this prevents the geometry around the masses to be of the Schwarzschild type and leads to the introduction of a cosmological constant. The study of the extension of the Kottler space-time shows that there exists a non-static solution consisting of two static regions surrounding the masses that match a Kantowski-Sachs expanding region on the cosmological horizon. The comparison with a Swiss-Cheese construction is also discussed.

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I. STATICITY, SPHERICAL SYMMETRY, AND EXPANSION

A key issue as regards the nature of the universe is how the smooth expanding large-scale universe is constructed from small-scale extremely inhomogeneous domains that are essentially static. The space-time in the Solar system for example is static to a very good approximation, but it is very inhomogeneous ($\delta\rho/\rho \simeq 10^{+30}$). How can one construct an expanding universe solution by gluing together large numbers of such quasi-static domains? The static space-time regions representing local masses and the vacuum regions around stars are of course structured into galaxies, clusters of galaxies, and large scale structures such as walls and voids; the same essential problem arises if we consider how they in turn can be assembled into an expanding universe. For simplicity and analytic clarity, we will assume that:

1. the component massive entities are exactly spherically symmetric;
2. these entities, whether we think of them as stars, galaxies, or galaxy clusters, are embedded in an exact vacuum solution of the Einstein field equations (possibly with a non-zero cosmological constant, Λ);
3. they are far enough apart from each other that the vacuum space-time around each of them is also locally spherically symmetric.

Given the assumption of local spherical symmetry around each mass, it is crucial that Birkhoff's theorem [1, 2] will apply:

4. the vacuum solution around each mass will necessarily be locally static.

Thus we do not have to assume local staticity which follows from the other previous assumptions. The local exterior space-time (outside the massive entities) will thus locally be either a Schwarzschild solution (if $\Lambda = 0$) or a Kottler solution (if $\Lambda \neq 0$). So we can ask: how can one join exact Schwarzschild or Kottler solutions together to form an expanding universe?

A. The Lindquist-Wheeler (L-W) solution

This issue was tackled in a very innovative paper by Lindquist and Wheeler a long time ago [3], using a Schwarzschild cell method to model an expanding universe with closed spatial sections (having the topology of a 3-sphere, S^3). For simplicity, they used a regular lattice, which allows a very limited set of possibilities. Considering N vertices, they state that “every vertex can be equidistant from its neighbours only when $N = 5, 8, 16, 120$, or 600 ”, which corresponds to the most homogeneous topologies of the 3-sphere [4]. They then derived equations of motion for the expanding universe from junction conditions between the cells. This work has been recently revisited in an interesting way in Refs. [5, 6].

This approach is quite different conceptually from the commonly used “Swiss-cheese” construction [7, 8]. In that case, one *starts off* with a Friedmann-Lemaître spatially homogeneous and isotropic geometry, and then cuts out spherical “vacuoles” within which individual masses are embedded. These masses are thus contained in vacua within a spatially homogeneous fluid-filled cosmos. In the Lindquist-Wheeler approach on the other hand, one starts off with the inhomogeneous vacuoles alone, and then glues them together to construct

an emergent Friedmann-Lemaître geometry when averaged on large scales. There is no fluid filling the space-time; rather (as in the case of kinetic theory) fluid-like behaviour emerges on large scales when one coarse-grains over the detailed structure. This is a far more fundamental approach to the study of the relation between locally static inhomogeneity and a globally expanding universe. However, this method is not strictly self-consistent in that the gravitational fields of the neighboring particles would in fact deform the field in the neighborhood of each cell's vertex, thereby resulting in an approximate rather than exact spherically symmetric space-time region (the real solution will be locally a bit anisotropic about each vertex). Nevertheless it is an acceptable approximation, and is a very useful approach to tackle the issue raised here.

B. Two-mass L-W solutions

In order to understand the situation better, we investigate the simplest possible such solution with isolated spherically symmetric massive objects embedded in a vacuum with S^3 spatial topology: namely the case with $N = 2$, which Lindquist and Wheeler did not consider [3]. With the two masses at antipodal points of the spatial sections, the space-time remains exactly spherically symmetric about each mass - the approximation comment made above no longer applies and an exact solution to the problem can be found. The exact spherical symmetry around each mass ensures that Birkhoff's theorem applies and the space-time is locally static. Setting off radially from either one of the masses, in order to have an S^3 spatial topology, the area of the surrounding 2-spheres $S^2(r)$ at proper distance r from the centre of symmetry must reach a maximum and then decrease to zero at the antipodal point: indeed we would expect to have two identical static solutions (if the 2 masses are equal) back to back, joined at the equator of the S^3 -spatial section.

Two striking results now emerge (as will be shown in details below). Firstly, if we want an exterior solution of this kind, we need a positive cosmological constant; $\Lambda > 0$. Hence the two local vacuum solutions to be joined together are Kottler space-times [9, 10] rather than Schwarzschild solutions, filled in by regular static bodies (e.g. stars) at their centers, so there is no horizon near the centre of symmetry. Let us call this a Kottler exterior solution.

Secondly, we first show that Israel junction conditions [11, 12] impose that the two space-times can only be joined on an horizon or a surface of maximum area, where one of the metric components vanishes. However, after constructing a coordinate system regular on the horizon, we show that the matching cannot be performed on the horizon. Gluing the two pieces together on a surface of maximum area back to back would result in a static configuration, but this is also shown to be impossible as no coordinate system can regularly cover the neighbourhood of the surface of maximum area. This prevents the existence of a static space-time analog to the Einstein-Static model with the mass concentrated into two antipodal compact objects. Instead, we find that the only solution contains two static regions matched across their null horizons to a pair of Kantowski-Sachs expanding and contracting universes; the expansion of the universe is possible because of the existence of these spatially homogeneous but anisotropic expanding regions. This solution is obtained by considering the global structure of the exterior Kottler space-time, which we show is essentially the same as that of the maximal de Sitter hyperboloid [2, 13].

Surprisingly, this is again rather like the Swiss-Cheese models, but now with the static domains embedded across null horizons (instead of space-like hypersurfaces in the case of the Swiss-Cheese). Each static and each expanding domain covers only part of the resulting

maximal Kottler exterior two-mass solution, in a way exactly analogous to what occurs in the maximally extended de Sitter solution. Furthermore, this shows that a globally static solution can exist only at the price of introducing a (unphysical) surface layer. A globally non-singular 2-mass LW space-time does indeed exist, but it has horizons separating spatially inhomogeneous static domains centred on the two masses from spatially homogeneous time-evolving region. It is this expanding universe domain that allows the two masses to move away from each other, and so is the reason this universe model can expand despite the static nature of space-time near each embedded massive object.

The rest of the text will establish these results and is organized as follows: section II is a quick summary of the statements of Birkhoff's theorem; section III will present the results obtained when one tries to glue together two spherically symmetric static space-times; section IV will discuss the results in a Swiss-cheese configuration and comment on the analogy with the previous gluing approach; finally section V will be a discussion of the results and their implications.

II. BIRKHOFF'S THEOREM

Birkhoff's theorem stands for the crucial result that a vacuum spherically symmetric space-time domain is necessarily either static or spatially homogeneous. This is a local result, valid even if Λ is non-zero. In brief, the proof (see Ref. [1] and also Appendix B of Ref. [2] and Refs. [14, 15]) can be summarized by considering a general spherically symmetric space-time with metric

$$ds^2 = -A(t, \chi)dt^2 + B(t, \chi)d\chi^2 + \chi^2 d\Omega^2, \quad (2.1)$$

(without loss of generality, we choose χ as an area coordinate). First, as long as the stress-energy tensor components of the source satisfy $T_{t\chi} = 0$, the Einstein equations imply that $\partial_t B = 0$. This implies that $B = B(\chi)$, which is the first part of the Birkhoff theorem. Then, the combination $G_t^t - G_\chi^\chi = 0$, that holds for a vacuum as well as a cosmological constant, implies that $B'A + BA' = 0$. It follows that A can be chosen so that $A = A(\chi)$ and $A(\chi)B(\chi) = K = 1$, so that

$$ds^2 = -A(\chi)dt^2 + A^{-1}(\chi)d\chi^2 + \chi^2 d\Omega^2. \quad (2.2)$$

The metric is static if $A > 0$ (surfaces of constant t are spacelike) and spatially homogeneous if $A < 0$ (surfaces of constant t are timelike).

Thus, the existence of an extra symmetry for the metric derives from its spherical symmetric nature and the fact that $T_{t\chi} = 0$ and $T_t^t - T_\chi^\chi = 0$, as is valid in the exterior region surrounding a point mass or an interior static solution (such as the Schwarzschild interior solution). In the latter case, which is the one of interest to us, by continuity the exterior region close to the static spherically symmetric object represented by the solution is chosen to be static rather than spatially homogeneous.

III. GLUING SPHERICALLY-SYMMETRIC SPACE-TIMES

A. Case of a spherically-symmetric static boundary

Let us consider two space-times with metric (2.2) and coordinate systems $(t_{\pm}, r_{\pm}, \theta, \varphi)$ in each of the two regions so that the metrics take the form

$$ds_{\pm}^2 = -A(r_{\pm})dt_{\pm}^2 + A^{-1}(r_{\pm})dr_{\pm}^2 + r_{\pm}^2 d\Omega^2. \quad (3.1)$$

We glue these space-times together on spacelike hypersurfaces defined by $\Sigma_{\pm} = \{r_{\pm} = R_{\pm}\}$ with R_{\pm} constants, assuming we identify the angular coordinates $\theta_{\pm} = \theta$ and $\varphi_{\pm} = \varphi$ (see Figure 1). The two normal vectors, chosen so that they are continuous across the junction ($n_{\mu}^{(+)}$ points out of its domain and $n_{\mu}^{(-)}$ into its domain), are given by $n_{\mu}^{(\pm)} = \pm A_{\pm}^{-1/2} \delta_{\mu}^{r_{\pm}}$. The induced metrics and extrinsic curvatures are thus given by

$$\gamma_{ab}^{(\pm)} dx^a dx^b = -A_{\pm} dt_{\pm}^2 + R_{\pm}^2 d\Omega^2, \quad (3.2)$$

$$K_{ab}^{(\pm)} dx^a dx^b = \pm \sqrt{A_{\pm}} \left(-\frac{1}{2} A'_{\pm} dt_{\pm}^2 + R_{\pm} d\Omega^2 \right). \quad (3.3)$$

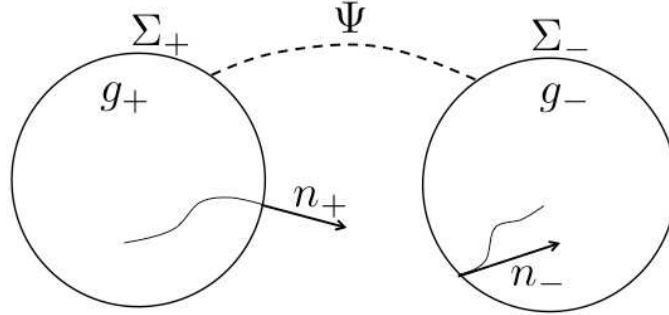


FIG. 1. Geometry of the matching between two space-times with metrics g_{\pm} and boundaries Σ_{\pm} that are identified via the mapping Ψ . n_{\pm} are the vectors normal to the hypersurfaces Σ_{\pm} .

It follows from Eq. (3.2), using the standard Israel junction conditions [11, 12, 16], that the continuity of the induced metric imposes that $R_{+} = R_{-} \equiv R$ and that $A_{+}(R)dt_{+}^2 = A_{-}(R)dt_{-}^2$. It then follows from Eq. (3.3) that the extrinsic curvature can be continuous only if $A_{\pm}(R) = 0$, which occurs at a horizon. In conclusion, *only on a horizon ($A = 0$) may two spherically symmetric static space-times possibly be glued back to back.*

We cannot use the Schwarzschild coordinates to perform the matching at the horizon because this coordinates system is singular there. Let us assume that \hat{r} is such that $A(\hat{r}) = 0$ and let us Taylor expand A in a neighborhood of \hat{r} as $A(r) = \lambda \times (r - \hat{r})$ with $\lambda \equiv A'(\hat{r}) \neq 0$. Using the coordinate r_{*} defined by $dr_{*}/dr = A^{-1}(r)$ we deduce that $r_{*} \simeq \lambda^{-1} \log(r - \hat{r})$ in a neighborhood of \hat{r} . Introducing the null-coordinates $u = t + r_{*}$ and $v = t - r_{*}$ and the relabelling $U = \exp(\lambda u/2)$ and $V = \exp(-\lambda v/2)$, the metric in the neighborhood of \hat{r} takes the form $ds^2 = \frac{4}{\lambda} dU dV + r^2 d\Omega^2$, and is regular in $r = \hat{r}$, that is on $UV = \exp(\lambda r_{*})$. r is

defined in such a way that $4\pi r^2$ is the surface area of a 2-sphere of radius r , which defines it uniquely.

Now, we can study the matching between two space-times

$$ds_{\pm}^2 = \frac{4}{\lambda_{\pm}} dU_{\pm} dV_{\pm} + r_{\pm}^2 d\Omega^2. \quad (3.4)$$

Any spherically symmetric spacelike hypersurface $\{r_{\pm} = R_{\pm}\}$ corresponds to a hypersurface of equation $\{U_{\pm} V_{\pm} = \text{const.}\}$ so that the normal unit spacelike vector is

$$n_{\mu}^{(\pm)} = \frac{\pm 1}{\sqrt{\lambda_{\pm} U_{\pm} V_{\pm}}} (V_{\pm} \delta_{\mu}^{U_{\pm}} + U_{\pm} \delta_{\mu}^{V_{\pm}}).$$

The induced metric is then given, as previously, by

$$\gamma_{ab}^{(\pm)} dx^a dx^b = -\frac{1}{\lambda_{\pm} U_{\pm} V_{\pm}} (V_{\pm} dU_{\pm} - U_{\pm} dV_{\pm})^2 + R_{\pm}^2 d\Omega^2$$

so that its continuity implies, as previously, $R_+ = R_- \equiv R$. Now, the continuity of the extrinsic curvature implies that $K_{\theta\theta}^{(\pm)} = -\Gamma_{\theta\theta}^{\alpha} n_{\alpha}^{(\pm)}$ is continuous, which is impossible. This shows that *the back to back matching of two static solutions cannot be performed on the horizon $A(r) = 0$.*

Thus, it is *impossible* to glue two spherically symmetric static space-times on a spherically symmetric and static boundary, even if the boundary is a horizon.

B. Case of a dynamical boundary

In the previous argument, we required the gluing surface to be a constant r hypersurface; in order to check if this is a restrictive assumption, let us generalize to the situation in which the boundary is moving, i.e. $R_{\pm} = R_{\pm}(t_{\pm})$. The two matching hypersurfaces are defined by $\Sigma_{\pm} = \{r_{\pm} - R_{\pm}(t_{\pm}) = \text{const}\}$, that we assume to be spacelike (we separately show that the null-surface case is indeed possible, see below). The two normal vectors are then given by

$$n_{\mu}^{(\pm)} = \frac{\pm 1}{\sqrt{A_{\pm} - \dot{R}_{\pm}^2/A_{\pm}}} (-\dot{R}_{\pm} \delta_{\mu}^{t_{\pm}} + \delta_{\mu}^{r_{\pm}}),$$

where $\dot{R}_{\pm} = dR_{\pm}/dt_{\pm}$. The induced metrics and extrinsic curvatures are given by

$$\gamma_{ab}^{(\pm)} dx^a dx^b = -\frac{A_{\pm}^2 - \dot{R}_{\pm}^2}{A_{\pm}} dt_{\pm}^2 + R_{\pm}^2 d\Omega^2 \quad (3.5)$$

and

$$K_{ab}^{(\pm)} dx^a dx^b = \frac{\pm 1}{A_{\pm} \sqrt{A_{\pm} - \frac{\dot{R}_{\pm}^2}{A_{\pm}}}} \left[\frac{1}{2} \left(3\dot{R}_{\pm}^2 A'_{\pm} - 2\ddot{R}_{\pm} A_{\pm} - A_{\pm}^2 A'_{\pm} \right) dt_{\pm}^2 + R_{\pm} A_{\pm}^2 d\Omega^2 \right] \quad (3.6)$$

using that on Σ_{\pm} , $dr_{\pm} = \dot{R}_{\pm} dt_{\pm}$. As R_{\pm} and A_{\pm} are both positive functions, it is impossible to have continuity of $K_{\theta\theta}$ unless $A_{\pm}[R_{\pm}(t_{\pm})] = 0$ for all t_{\pm} . This imposes that $\dot{R}_{\pm} = 0$, and thus, a configuration with $\dot{R}_{\pm} \neq 0$ cannot be constructed if the join surfaces are timelike.

C. Conclusions

To summarize, two spherically symmetric static solutions cannot be glued together back-to-back to obtain a static space-time with the spatial topology of a 3-sphere. In particular, this implies that we cannot glue two Schwarzschild exterior solutions together to get a 2-mass space-time with S^3 topology. This can be possible only at the expense of introducing a surface layer, which we do not want to consider.

IV. TWO MASS SOLUTION

We now show that a spatially closed space-time with two masses the spatial size of which is larger than their Schwarzschild horizon can be constructed using two copies of the exterior Kottler space-time [9, 10], which is the generalisation of the Schwarzschild solution to incorporate a non-vanishing cosmological constant. This solution has two horizons, a black-hole horizon at $r = r_b$ similar to the Schwarzschild horizon and a cosmological horizon at $r = r_c$ similar to the de Sitter horizon.

As we shall see, a solution with a S^3 spatial topology with two antipodal masses can be constructed but with an expanding domain in between the two static domains, as will become clear once we construct the Penrose-Carter diagram for the solution.

A. Kottler space-time

The Kottler solution [9, 10] (for a summary see Ref. [17]) is the extension of the Schwarzschild solution to include a cosmological constant,

$$ds^2 = -A(r)dt^2 + A^{-1}(r)dr^2 + r^2d\Omega^2, \quad (4.1)$$

$$A(r) = 1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}. \quad (4.2)$$

It is easy to check that the Killing vector

$$\xi^\mu = \delta_0^\mu \quad (4.3)$$

has norm $g_{\mu\nu}\xi^\mu\xi^\nu = A(r)$ and is thus timelike as long as $A > 0$. We thus have two cases:

- If $9(GM)^2\Lambda > 1$, A is negative for $r > 0$ so that ξ^μ is spacelike and the space-time contains no static region but is spatially homogeneous. We exclude this case as it does not allow a static central body.
- If $9(GM)^2\Lambda < 1$, A is positive for r between r_b and $r_c > r_b$ which correspond respectively to the black-hole and cosmological horizons. They can be found to be given by

$$r_c = \frac{2}{\sqrt{\Lambda}} \cos\left(\frac{\psi}{3} + \frac{\pi}{3}\right), \quad r_b = \frac{2}{\sqrt{\Lambda}} \cos\left(\frac{\psi}{3} - \frac{\pi}{3}\right), \quad \cos\psi = 3GM\sqrt{\Lambda} \quad (4.4)$$

so that we have

$$2GM < r_b < 3GM < \frac{1}{\sqrt{\Lambda}} < r_c < \frac{3}{\sqrt{\Lambda}}.$$

The space-time is thus static in the region $r_b < r < r_c$. It is clear that the third root is negative since $A = -(\Lambda/3r)(r - r_b)(r - r_c)(r + r_b + r_c)$. $r = r_b$ and $r = r_c$ are Killing horizons since ξ vanishes on these hypersurfaces. One can check that for both values, $A'(r) \neq 0$ (the horizons are not degenerate) and $A''(r) < 0$.

We consider solutions of this second type where an interior solution representing a spherically symmetric body occupies the region $0 < r < R_M$. This requires that $R_M > r_b$, so the spatially homogeneous vacuum region for $0 < r < r_b$ is filled in by the body; hence there is no horizon at $r = r_b$. We require that $R_M < r_c$ (there is some vacuum region around the central body). Such a solution will be called a *Kottler exterior solution*.

The metric can be conveniently rewritten in terms of the radial coordinate r_* defined by

$$dr_* \equiv \frac{dr}{A(r)} \quad (4.5)$$

that is explicitly given by

$$r_* = \frac{3}{\Lambda} \left[\frac{r_b \ln(r - r_b)}{(r_c - r_b)(r_b - r_s)} - \frac{r_c \ln(r - r_c)}{(r_c - r_b)(r_c - r_s)} + \frac{r_s \ln(r - r_s)}{(r_b - r_s)(r_c - r_s)} \right] \quad (4.6)$$

with $r_s \equiv -(r_b + r_c) = -2/\sqrt{\Lambda} \cos(\psi/3)$.

B. Penrose-Carter diagram

To explicitly describe this solution, let us construct its Penrose-Carter diagram (see Ref. [2]). Obviously, the metric can be rewritten using the ingoing Eddington-Finkelstein coordinates as

$$ds^2 = -Adu^2 + 2dudr + r^2 d\Omega^2, \quad (4.7)$$

with $du = dt + \frac{1}{A}dr = dt + dr_*$ or the outgoing Eddington-Finkelstein coordinates as

$$ds^2 = -Adv^2 - 2dvdr + r^2 d\Omega^2, \quad (4.8)$$

with $dv = dt - \frac{1}{A}dr = dt - dr_*$. It is clear under the first form that ∂_u is a Killing vector [18]. When $A > 0$, it is spacelike and one recovers the Kottler solution under its static form (set $r = v$ and $dt = du - dv/A$) while when $A < 0$ the Killing vector is timelike and one gets a homogeneous solution (set $t = v$ and $dr = du - dv/A$). Using the null-coordinates (u, v) leads to the usual form

$$ds^2 = -Adudv + r^2 d\Omega^2, \quad (4.9)$$

which remains pathological when $A = 0$. As usual with horizon, the surfaces of constant u or constant v (resp. ingoing/outgoing null geodesics) are geometrically well defined but their labelling is not on the horizon.

When focusing on the exterior Kottler solution, the only horizon of interest is located in $r = r_c$ (remind that $r_s < 0 < r_b < R_M < r_c$). Introducing

$$U = \exp \left[-\alpha \frac{u}{2} \right], \quad V = \exp \left[\alpha \frac{v}{2} \right],$$

with $\alpha = (\Lambda/3)(r_c - r_b)(r_c - r_s)/r_c$, the metric becomes

$$ds^2 = -BdUdV + r^2d\Omega^2, \quad (4.10)$$

with $B \equiv -(4A/\alpha^2) \exp(\alpha r_*)$. Given the expression (4.6), it is explicitly given

$$B = \frac{\Lambda}{3r} \frac{4}{\alpha^2} (r - r_b)^{1 + \frac{r_b(r_c - r_s)}{r_c(r_b - r_s)}} (r - r_s)^{1 + \frac{r_s(r_c - r_b)}{r_c(r_b - r_s)}},$$

which does not vanish for $r \in [R_M, +\infty[$. This is similar to the standard analysis to construct a coordinate system regular on the horizon of a Schwarzschild black-hole (see e.g. Ref. [19]). We refer to Refs. [20] for the definition of a maximally extended map that covers the Kottler solutions and that generalizes the maps introduced in Refs. [21, 22] for Schwarzschild and Reissner-Nordström. For a detailed description of the global structure and horizons of the Kottler space-time see Refs. [23, 24].

From this latter form, we can deduce the Penrose-Carter diagram depicted on Fig. 2. The Penrose-Carter diagram of our exterior solution with two identical masses is only a subset of the general diagram and is depicted on Fig. 3. It is obtained by gluing another copy of the space-time described on the right-hand side of figure 2 and by cutting the two static regions at a given radius $R_M > r_b$ (represented by the thick lines) to restrict to the exterior solution of interest. We see that it exhibits two static regions around each mass extending up to the cosmological horizons (at $r = r_c$). The two horizons cannot be glued together and there must exist a homogeneous region in between them. In this homogeneous region connecting the two static regions, the r coordinate is effectively a timelike coordinate whereas t is a radial coordinate. Note that the two massive bodies are causally disconnected: light emitted in the static region surrounding one of the masses will eventually cross the first horizon, reach the expanding region and attain \mathcal{J}^+ (that is timelike) but will never cross the second horizon. Note also that the Killing vector cannot be timelike everywhere, since the horizon is a null surface. This is the reason why there is no analog of the Einstein static Universe with the masses concentrated in two antipodal compact objects. Let us also note that similar diagrams have been constructed in Ref. [25] while studying constant tension stars and hybrid stars (i.e. having an interior zone with negative pressure and an infinitely thin outermost layer with positive pressure and energy density) in a Schwarzschild spacetime.

C. Embedding

This general construction can be understood by an embedding in a 5-dimensional Minkowski space-time [26, 27]. Indeed, the exact form of the embedding cannot be determined analytically but assuming the masses are such that $r_b \ll r_c$, the structure of the space-times on cosmological scales are closed to the one of a de Sitter space-time. Nevertheless, the worldlines of the two masses do not have the symmetries of a de Sitter space-time and therefore induces a preferred slicing of space-time; hence resulting in a preferred time direction.

The expanding region has a metric given by

$$ds^2 = -\frac{1}{A(t)}dt^2 + A(t)dr^2 + t^2d\Omega^2. \quad (4.11)$$

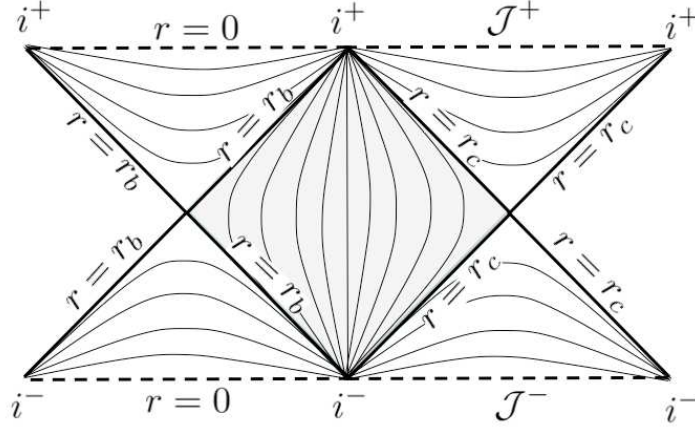


FIG. 2. Penrose-Carter diagram of a Kottler space-time. Thin lines represent constant r hypersurfaces. The space-time is static in the central lozenge region and homogeneous in the other regions.

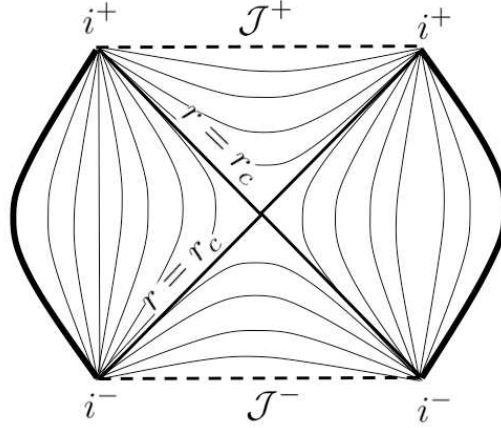


FIG. 3. Exterior two-mass Kottler solution.

Defining $d\tau = dt/\sqrt{A(t)}$ and $B^2(\tau) = A(t(\tau))$, it takes the form

$$ds^2 = -d\tau^2 + B^2(\tau)dr^2 + t^2(\tau)d\Omega^2. \quad (4.12)$$

This anisotropic expanding region is the interior solution of a black-hole (see e.g. [28]) in the vacuum case, but is of the Kantowski-Sachs form when filled with a fluid (and possibly a cosmological constant) [29]. This is thus another kind of Swiss cheese model but within an anisotropic vacuum space-time in the expanding region.

Taking horizontal sections at different times, we get S^3 spatial slicings that contain back to back copies of a massive object surrounded by a static vacuum that is in turn surrounded by

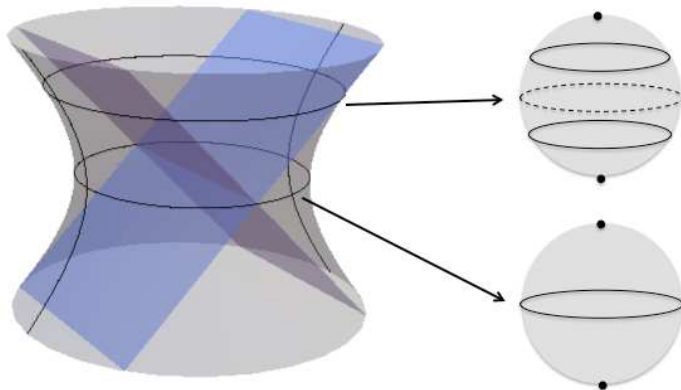


FIG. 4. Embedding of the solution in a 5-dimensional space-time. The hyperboloid is a 4-dimensional space-time. Each circle is a spacelike section which has the topology of the 3-sphere. The two lines represent the worldlines of the masses. For each slice, the space has the topology of a 3-sphere with a mass at the North pole and a mass at the South pole. We have depicted the horizon (solid line).

an expanding universe domain that has a cross sectional area that increases to a maximum, where it is matched to an identical solution in back to back fashion (see the inset diagram in Fig 4). As time progresses, the size of the expanding universe section decreases from most of the spatial section (at large negative times) to zero and then increases again to most of the spatial sections (at large positive times). This is like the Wheeler analysis of the time evolution of the throat in a black hole universe.

D. Conclusion

This gives a two-mass version of the de Sitter expanding universe. It can be envisioned as two world lines added antipodally to the maximal $S^3 \times R$ de Sitter hyperboloid. If these are of very small mass, by continuity the global structure [2, 13] will remain unchanged. Indeed this is the structure of the two-mass exterior Kottler solution, as can be seen by filling in the two Schwarzschild horizons in the maximal Kottler space-time.

V. THE SWISS-CHEESE APPROACH

A standard method to embed a single mass in an expanding space-time is to use the Einstein-Strauss method [7, 30] to construct a Swiss-cheese model [31]. We thus consider the two space-times

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2 d\Omega^2, \quad (5.1)$$

and

$$ds^2 = -dT^2 + a^2(T) (d\chi^2 + \sin^2 \chi d\Omega^2), \quad (5.2)$$

describing the geometry of a Kottler space-time and of a Friedmann-Lemaître (FL) universe with spherical spatial sections having chosen units such that the comoving curvature radius $R_c = 1$. We decide to glue these two space-times on a constant $r = r_0(t)$ hypersurface in the Kottler space-time and on a $\chi = \chi_*$ hypersurface. In the FL region, the normal is given by $n_\mu^{(\text{FL})} = \delta_\mu^\chi / a$ so that

$$\gamma_{ab}^{(\text{FL})} dx^a dx^b = -dT^2 + a^2(T) \sin^2 \chi_* d\Omega^2, \quad K_{ab}^{(\text{FL})} dx^a dx^b = -a(T) \sin \chi_* \cos \chi_* d\Omega^2. \quad (5.3)$$

We stress that we note $\dot{a} = da/dT$ and $\dot{r}_0 = dr_0/dt$. Since

$$\gamma_{ab}^{(\text{K})} dx^a dx^b = -\frac{A^2 - \dot{r}_0^2}{A} dt^2 + r_0^2 d\Omega^2, \quad (5.4)$$

the continuity of the induced metric implies that

$$r_0(t) = a(T) \sin \chi_*, \quad \frac{dT}{dt} = \sqrt{\frac{A^2[r_0(t)] - \dot{r}_0^2(t)}{A[r_0(t)]}}, \quad (5.5)$$

which defines the worldsheet of the hypersurface on which we match in the Kottler region and the relation between the times in both regions.

Now in the FL region, the scale factor must satisfy the Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{1}{a^2} + \frac{\Lambda}{3}, \quad (5.6)$$

$\rho = \rho_0(a_0/a)^3$ being the energy density of a pressureless fluid. The continuity of the extrinsic curvature is achieved only if the cosmological constant is the same in the two space-times and if

$$M = \frac{4\pi}{3} \rho a^3 \sin^3 \chi_*. \quad (5.7)$$

It follows that

$$\left(\frac{dT}{dt}\right)^2 \cos^2 \chi_* = A^2[a(T) \sin \chi_*]. \quad (5.8)$$

Hence given a FL space-time with pressureless matter and a cosmological constant, we know $a(T)$ and we can insert a spherical region of radius $\sin \chi_*$ which contains a constant mass $M = (4\pi \rho a^3 \sin^3 \chi_*)/3$ at its center and has the Kottler geometry. Then Eq. (5.8) gives the relation between the coordinate times of the two space-times.

The limit $\chi_* \rightarrow \pi/2$ is such that $A \rightarrow 0$, i.e. the mass becomes such that the equatorial 2-sphere is also the cosmological horizon of the Kottler space-time.

With such a solution, we can indeed insert two Kottler holes in the FL solution and make their size increase (see Fig. 5-left). Indeed, if they are not antipodal then the two boundaries (which are 2-sphere) will not match. If they are, we can increase their size until they join, at which point they will contain, by construction the mass. The induce metric on the boundary will be continuous but, as can be seen from Fig. 5-right, the extrinsic curvature will not be continuous (in particular because the unit normal vector is going in the FL spacetime for one hole and outside for the other). When the two holes join, they are glued on their horizon and we are back to the situation of Fig. 1. We thus recover the solution of the first section.

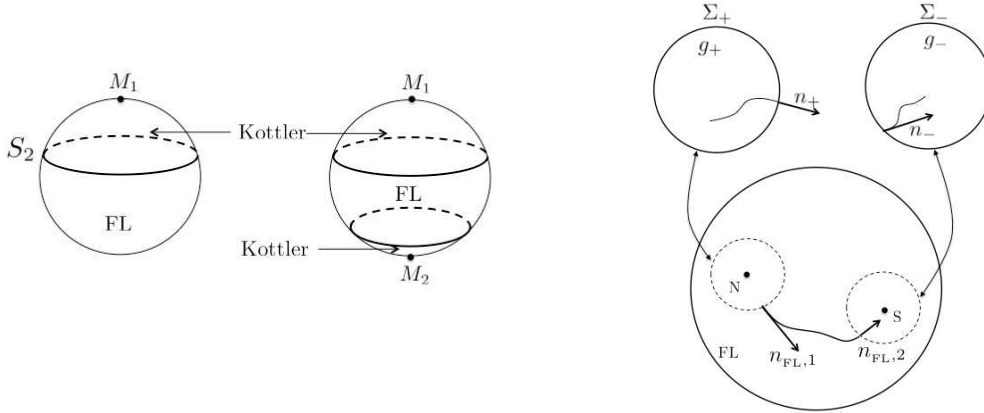


FIG. 5. (Left) Construction of two Kottler holes in a FL space-time with spherical spatial sections. The mass inside each hole depends on the size of the hole. (Right) Consider two Kottler holes (respectively with boundaries Σ_+ and Σ_- and associated normal vectors n_+ and n_-) inside a FL spacetime (with $n_{\text{FL},1}$ and $n_{\text{FL},2}$ for the normal vectors associated with the two boundaries). The two holes can be extended in order to join smoothly only if their centers are antipodal. In that limit $n_{\text{FL},1} \rightarrow n_{\text{FL},2}$. The two holes thus contain the same mass but there remains a surface layer related to the jump of the extrinsic curvature.

VI. DISCUSSION

The LW models are significant tools in elucidating the ongoing discussion on averaging in cosmology, and the relation between small scale inhomogeneous models and large scale spatially homogeneous models (see e.g. [5, 6, 33–35]). This could provide a better understanding of the emergence of a fluid-like description of the Universe on large scales from a more realistic local clumpy distribution of matter. In the line of this program, we have analyzed the geometrical structure of an exact two-mass singularity free version of the LW models: an expanding model which is the analogue of the maximal de Sitter hyperboloid once it has been embedded in a 5 dimensional space-time. This resolves the paradox raised in the introduction to this paper: how can locally static domains be glued together to give an expanding universe? Indeed, the solution presented here is locally static around each compact object but not globally so even if the conditions of the Birkhoff theorem still holds (spherical symmetry around each mass and vacuum + cosmological constant solution of Einstein equations). This is different from a de Sitter space-time, which is locally static everywhere: nothing intrinsic to the de Sitter space-time identifies any particular null surface as the horizon, and there are local timelike and null Killing vector fields at every point. The 2-mass exterior Kottler solution, on the other hand, is not locally static in the region across the horizon: as in the case of the Schwarzschild solution (inside its horizon), there is no timelike Killing vector field there, and the horizon is uniquely defined locally by existence of a null Killing vector field.

We have also shown that it is not possible to construct a globally static solution that is the analogue of the Einstein Static universe (a timelike Killing vector field occurs at every point), but with the mass concentrated in two antipodal massive bodies. This would require the introduction of an unphysical surface layer. We have shown that the expanding region

can either be filled with a cosmological constant and has a Kantowski-Sachs geometry, or by a pressureless fluid and a cosmological constant. In that latter case, the geometry of the expanding region can be of the FL type.

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